THE MATHEMATICAL PARADOXES FOR THE FLOW OF A VISCOPLASTIC FILM IN COMPLEX GEOMETRIES

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Theoretical and experimental fluid mechanics is the realm of several well-known paradoxes as well as of endless debates about the correctness of some mathematical models, commonly accepted as extremely useful to describe the complexity of several fluid phenomena. Most of classical paradoxes have been intensively investigated for decades or even centuries. Besides being interesting in themselves, paradoxes have always been a fundamental trust towards new ideas or methods (or both).

Among the most known ones we find the famous *d'Alembert's paradox* (if irrotational inviscid flow theory is used, the drug on a steady body hit by an infinite stream is zero), and the *Stokes' paradox* (the Stokes approximation of the full Navier-Stokes equation at very low Reynolds number does not have a bounded solution past an infinitely long cylinder of any shape hit by an infinite stream). Less famous ones are, for example, the *Grey's paradox* (unexplainable high-speed performance of dolphins: some aquatic animals such as tuna and dolphin can swim at a speed higher than the value estimated from both of the muscle power and the frictional drag received by the fluid flow around the animals), the *Sternberg-Koiter paradox* (as revealed by Moffatt in problems on the creeping motion of a fluid in an angular region) or that one we are going to focus on today, namely the so-called *"lubrication paradox for yield stress fluids"*: when the Bingham model is used, for example, to describe the evolution of a fluid with a yield stress behaviour in a thin layer or a narrow pipe with wavy boundary, something goes wrong and the model fails. The same happens with other non elementary geometries of great interest in applications.

Analyses of visco-plastic fluid flows in complex geometries of small aspect ratio have a relatively long history. The main question is whether a true unyielded plug region exists or not for a yield stress fluid in these cases. Lipscomb and Denn (see [12]) were probably the first to argue that, in the case of squeeze flow, a true rigid plug regions should not exist. In practice the paradox comes from the observation that, when performing classical lubrication scaling, the unyielded plug region moves with a speed that varies along the principal direction of the motion, *contradicting the hypothesis that the plug behaves as a rigid body*.

After Lipscomb and Denn, many ways of overcoming such a physical paradox have been proposed by different authors. In [11, 12] a two-viscosity model was exploited to clear up the squeeze-flow paradox, with the Bingham model that can be recovered as a limiting case. Other "corrections" are possible, as the one used in [10] to model the evolution of lava domes, where the Bingham and Herschel-Bulkley model is "regularized" allowing for a weakly yielding of the plug.

In this lecture we review our recent contributions (see [1, 2, 3, 4, 5, 6, 7, 8]) concerning the lubrication paradox for yield stress fluids. Our approach, based on a paper of two of us (see [9]), allows to overcome the lubrication paradox in various complex geometries.

Keywords: *Bingham flow, lubrication paradox, multiscale problems*

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